Mathematical Challenges
for
High School Students

(useful for Math Olympiads, ISI, CMI ent.)

It's like asking why is Ludwig van Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.

Paul Erdos

Titles in this Series

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for
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(Useful for Math Olympiads, ISI, CMI ent.)

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Mathematics for All
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Dedicated
to my father
Bhudhar Chandra Roy

from whom
I learned my first
lesson of Mathematics

Math1089 is an engaging platform dedicated to exploring the fascinating world of mathematics. It offers a creative blend of mathematical concepts, patterns, puzzles, and problem-solving strategies aimed at fostering curiosity and a love for numbers among its audience. Whether you're a student, teacher, or math enthusiast, Math1089 provides innovative content, including number patterns, arithmetic wonders, geometric insights, and real-world applications of mathematics.

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A Fresh Start . . .

Mathematics is often described as the poetry of logical ideas, and nowhere is this more evident than in the realm of problem solving. Math Olympiads provide young learners with the opportunity to experience this poetry firsthand—challenging them to think beyond routine methods, explore multiple strategies, and discover the elegance hidden in mathematical reasoning.

This book is designed with the spirit of Olympiad Mathematics in mind. It goes beyond classroom exercises, offering non-routine and thought-provoking problems that demand creativity, persistence, and insight. The questions are carefully chosen to spark curiosity, encourage deeper understanding, and cultivate problem-solving habits essential for success in Olympiads as well as in higher mathematical pursuits.

Each problem is more than just a test of skill—it is an invitation to explore concepts, build connections, and appreciate the beauty of mathematics. Where possible, multiple approaches are highlighted to show that mathematics is not merely about finding answers, but about the joy of discovery.

This book is intended for students who aspire to excel in Math Olympiads, teachers who wish to enrich their classrooms with stimulating content, and anyone who enjoys the challenge of solving mathematical challenges. It is my hope that readers will not only sharpen their problem-solving abilities but also develop a lifelong love for mathematics.

Mathematics is a journey with endless paths, and Olympiad problems are some of the most exciting milestones along the way. May this book serve as a companion in your mathematical adventure, guiding you to think deeply, reason clearly, and embrace the true spirit of problem solving.

Of course, despite careful preparation, some errors may remain. If you happen to come across any inaccuracies, I would be deeply grateful if you could bring them to my attention. Kindly write to me at **`math1089.9801@gmail.com`**, so corrections can be made in future editions.

A heartfelt note of gratitude is owed to Avirupa and Adrija for the sacrifices they made during times I was unavailable to play or spend time with them. I also extend my sincere thanks to my family for their enduring patience and support throughout this journey.

Happy exploring!

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27/08/2025
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What is Where?

1 Problems

Mathematics is the science of the connection of magnitudes. Magnitude is anything that can be put equal or unequal to another thing. Two things are equal when in every assertion each may be replaced by the other.

Hermann Grassmann

P1. Given the values of x and y as

$$x = \frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + \cdots + \frac{2022^2}{4043}$$

and

$$y = \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \cdots + \frac{2021^2}{4043}.$$

Find the value of $x - y$.

P2. Simplify the following expression

$$\frac{(10^4 + 324)(22^4 + 324)(34^4 + 324)(46^4 + 324)}{(4^4 + 324)(16^4 + 324)(28^4 + 324)(40^4 + 324)}.$$

P3. Resolve 99899 into factors.

P6. Let

$$P=\frac{496^2+436^2+100^2+40^2-9^2-69^2-405^2-465^2}{62^3+248(29^2+13^2+7^2)}.$$

Find the value of $8P$.

P16. The present ages in years of two brothers A, B and their father C are three distinct positive integers a, b and c respectively. Suppose $\frac{b-1}{a-1}$ and $\frac{b+1}{a+1}$ are two consecutive integers, and $\frac{c-1}{b-1}$ and $\frac{c+1}{b+1}$ are two consecutive integers. If $a+b+c \le 150$, determine a, b and c.

P47. Each of the numbers $x_1, x_2, \dots, x_n$ equal to 1 or -1 and $S = x_1x_2x_3x_4 + x_2x_3x_4x_5 + \cdots + x_nx_1x_2x_3 = 0$. Prove that $4|n$.

P48. Prove that the sum

$$S=\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

where $n > 1$ cannot be an integer.

P74. If a, b, c are three positive real numbers such that $a+b \ge c$, show that $\min\{1, a\} + \min\{1, b\} \ge \min\{1, c\}$.

P75. Let $x_1, x_2, \dots, x_{100}$ satisfy

$$\sqrt{1+x_1}+\sqrt{1+x_2}+\cdots+\sqrt{1+x_{100}}=100\sqrt{1+\frac{1}{100}}$$

and

$$\sqrt{1-x_1}+\sqrt{1-x_2}+\cdots+\sqrt{1-x_{100}}=100\sqrt{1-\frac{1}{100}},$$

Find the value of $x_1+\cdots+x_{100}$.

P76. If all possible values of

$$M = \frac{|x+y|}{|x|+|y|} + \frac{|y+z|}{|y|+|z|} + \frac{|z+x|}{|z|+|x|}$$

where x, y, z are non-zero real numbers, lies in the interval $[a, b]$, find the value of $b - a$.

P81. If $\theta \in (e^{-\pi/2}, \ \pi/2)$, prove that $\cos(\log_e \theta) > \log_e(\cos\theta)$.

P90. Consider all pairs (x', y') of integers that satisfy the equation $|xy| + |x - y| = 1$. Let $D = \max\{x' + y'\}$ and $d = \min\{x' + y'\}$. Then find the value of $5(D - d)$.

P104. *Ackerman's function* is defined for non-negative integers n and k by the following equations:

(**1**) $f(0, n) = n + 1$

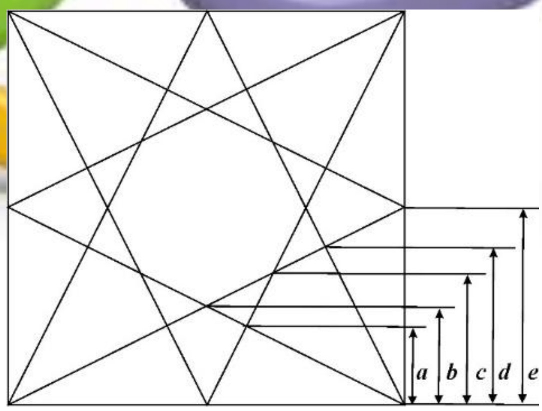
(**2**) $f(k, 0) = f(k - 1, 1)$

(**3**) $f(k + 1, n + 1) = f(k, f(k + 1, n))$.

Find the value of $f(2, 2)$.

P108. For $n \in \mathbb{N}$, let $I_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ and $J_n = \left(0, \frac{1}{n}\right]$. Prove that $I_{n+1} \subset I_n, J_{n+1} \subset J_n$ and $I_1 \cap I_2 \cap \ldots = \{0\}, J_1 \cap J_2 \cap \ldots = \phi$.

P111. Calculate the lengths a, b, c, d and e in terms of side of square. Assume that the given figure is symmetrical.



2 Solutions

Mathematics is on the artistic side a creation of new rhythms, orders, designs, harmonies, and on the knowledge side, is a systematic study of various rhythms, orders, designs and harmonies.

William L. Schaaf

S1. After rearranging, we find that $x - y$

$$= \left(\frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + \cdots + \frac{2022^2}{2020}\right) - \left(\frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \cdots + \frac{2021^2}{4043}\right)$$

$$= \frac{1^2}{1} + \left(\frac{2^2}{3} - \frac{1^2}{3}\right) + \left(\frac{3^2}{5} - \frac{2^2}{5}\right) + \cdots + \left(\frac{2022^2}{4043} - \frac{2021^2}{4043}\right)$$

$$= \frac{1^2}{1} + \left(\frac{2^2 - 1^2}{3}\right) + \left(\frac{3^2 - 2^2}{5}\right) + \cdots + \left(\frac{2022^2 - 2021^2}{4043}\right)$$

$$= \frac{1^2}{1} + \frac{(2+1)(2-1)}{3} + \cdots + \frac{(2022+2021)(2022-2021)}{4043}$$

$$= 1 + 1 + 1 + \cdots + 1$$

$$= 2022.$$

S3. Let $f(x) = 9x^4 + 9x^3 + 8x^2 + 9x + 9$. Then we have

$$\begin{aligned} f(x) &= x^2\left(9x^2 + 9x + 8 + \frac{9}{x} + \frac{9}{x^2}\right) \\ &= x^2\left[9\left(x^2 + \frac{1}{x^2}\right) + 9\left(x + \frac{1}{x}\right) + 8\right] \\ &= x^2\left[9\left(x + \frac{1}{x}\right)^2 - 18 + 9\left(x + \frac{1}{x}\right) + 8\right] \\ &= x^2\left[9\left(x + \frac{1}{x}\right)^2 + 9\left(x + \frac{1}{x}\right) - 10\right] \\ &= x^2\left[3\left(x + \frac{1}{x}\right) - 2\right]\left[3\left(x + \frac{1}{x}\right) + 5\right] \\ &= (3x^2 - 2x + 3)(3x^2 + 5x + 3) \\ &= g(x)h(x), \end{aligned}$$

say, where

$$g(x) = (3x^2 - 2x + 3) \text{ and } h(x) = (3x^2 + 5x + 3).$$

Therefore, $\quad 99899 = f(10) = g(10)h(10)$
$$= 283 \times 353.$$

S16. Without any loss of generality, assume that $a \le b$. Then,

$$\frac{b-1}{a-1} \ge \frac{b+1}{a+1}.$$

Let us assume that

$$\frac{b-1}{a-1} = L, \qquad \frac{b+1}{a+1} = L - 1$$

and

$$\frac{c-1}{b-1} = M, \qquad \frac{c+1}{b+1} = M - 1$$

From the first two relations:

$$b - 1 = L(a-1), b + 1 = (L-1)(a+1).$$

Solving for a, we get $a = 2L - 3$ and hence $b = 2L^2 - 4L + 1$.
Using the last two relations:

$$c - 1 = M(b-1), c + 1 = (M-1)(b+1).$$

Solving for b, we get $b = 2M - 3$ and hence $c = 2M^2 - 4M + 1$.
Thus, we have

$$2M - 3 = 2L^2 - 4L + 1 \text{ or } M = (L-1)^2 + 1.$$

Obviously, $L > 1$. If $L = 2$, we get $a = 2L - 3 = 1$. Then,

$$b = 1 + L(a-1) = 1,\ M = (b+3)/2 = 2 \text{ and } c = 1,$$

which is not possible.

Similarly, if $L = 3$, we get $a = 3$, $b = 7$ and $c = 31$.

If $L \geq 4$, then $m \geq 10$ and $c \geq 161$. But then, $a + b + c > 150$.

Thus, the only choice is $a = 3, b = 7$ and $c = 31$.

S47. Let

$$y_i = x_i x_{i+1} x_{i+2} x_{i+3} \text{ for } i = 1, 2, \dots, n,$$

where $x_{n+i} = x_i$. Given that

$$y_1 + y_2 + \cdots + y_n = 0.$$

Since each $x_i = 1$ or -1, we have $y_i = 1$ or -1 for each i (in any order). Assume that $y_i = 1$ for n_1 values of i and $y_i = -1$ for n_2 values of i.

Then $n_1 + n_2 = n$. Now

$$0 = y_1 + y_2 + \cdots + y_n = n_1(1) + n_2(-1)$$
$$\Rightarrow 0 = n_1 - n_2$$
$$\Rightarrow n_1 = n_2.$$

Hence, $n = 2n_1 = 2n_2$ and

$$y_1 y_2 \cdots y_n = (1)^{n_1} \times (-1)^{n_2} = (-1)^{n_2}.$$

But

$$\begin{aligned} y_1 y_2 \cdots y_n &= (x_1 x_2 x_3 x_4) \cdots (x_n x_1 x_2 x_3) \\ &= x_1^4 x_2^4 x_3^4 x_4^4 \\ &= 1 = (-1)^{n_2} \end{aligned}$$

proving that n_2 must be even.

Let $n_2 = 2p$ where $p \in \mathbb{Z}$.

Then $n = 2n_2 = 4p$, proving that $4|n$.

S48. To prove that S is not an integer, our task is to find an integer A such that the product $A \cdot S$ is not an integer.

Different numbers present in the denominators of S are $2, 3, \dots, n$.
There is only one number from among $2, 3, \dots, n$ that has the highest power of 2.
Let this highest power of 2 be 2^k and $2^k \leq n$.
Consider $B = 3 \cdot 5 \cdot 7 \cdots m$, the product of all odd numbers, where m is the greatest odd integer such that $m \leq n$. Let,

$$A = 2^{k-1}B = 2^{k-1}(3 \cdot 5 \cdots m) \in \mathbb{Z}.$$

Now

$$A \cdot S = \frac{2^{k-1}(3 \cdot 5 \cdots m)}{2} + \frac{2^{k-1}(3 \cdot 5 \cdots m)}{3} + \cdots + \frac{2^{k-1}(3 \cdot 5 \cdots m)}{2^k} + \cdots + \frac{2^{k-1}(3 \cdot 5 \cdots m)}{n}$$

All the terms in the above summation are integers other than

$$\frac{2^{k-1}(3 \cdot 5 \cdots m)}{2^k}.$$

Therefore, $A \cdot S \notin \mathbb{Z}$.
Since $A \in \mathbb{Z}$, we conclude that S is not an integer.

S75. We know the inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

Using this inequality for the numbers

$$\sqrt{1 + x_1}, \sqrt{1 + x_2}, \dots, \sqrt{1 + x_{100}}$$

and

$$\sqrt{1 - x_1}, \sqrt{1 - x_2}, \dots, \sqrt{1 - x_{100}},$$

we find that

$$\frac{\sqrt{1 + x_1} + \cdots + \sqrt{1 + x_{100}}}{100} \leq \sqrt{\frac{(1 + x_1) + \cdots + (1 + x_{100})}{100}}$$

$$\Rightarrow \sqrt{1 + \frac{1}{100}} \leq \sqrt{1 + \frac{(x_1 + \cdots + x_{100})}{100}}$$

$$\Rightarrow 1 + \frac{1}{100} \leq 1 + \frac{(x_1 + \cdots + x_{100})}{100}$$

$$\Rightarrow 1 \leq x_1 + \cdots + x_{100}$$

and similarly,

$$\frac{\sqrt{1 - x_1} + \cdots + \sqrt{1 - x_{100}}}{100} \leq \sqrt{\frac{(1 - x_1) + \cdots + (1 - x_{100})}{100}}$$

$$\Rightarrow \sqrt{1 - \frac{1}{100}} \leq \sqrt{1 - \frac{(x_1 + \cdots + x_{100})}{100}}$$

$$\Rightarrow 1 - \frac{1}{100} \leq 1 - \frac{(x_1 + \cdots + x_{100})}{100}$$

$$\Rightarrow -1 \leq -(x_1 + \cdots + x_{100})$$

$$\Rightarrow 1 \geq x_1 + \cdots + x_{100}$$

It follows that, $x_1 + \cdots + x_{100} = 1$.

S104. Now, repeated application of (**3**) yields

$$f(2,2) = f(1+1,1+1) = f\big(1, f(1+1,1)\big)$$
$$= f\big(1, f(1+1,0+1)\big) = f(1, f(1, f(2,0))).$$

Applying (**2**) and (**3**), we get

$f(2,0) = f(1,1) = f(0+1,0+1) = f(0, f(1,0))$.

Now, $f(1,0) = f(0,1)$ [by (**2**)] $= 1 + 1 = 2$, by (**1**).

Hence $f(2,0) = f(1,1) = f(0,2) = 2 + 1 = 3$,

so that $f(2,2) = f(1, f(1,3))$.

Also, $f(1,m) = f(0+1,(m-1)+1)$

$$= f\big(0, f(1, m-1)\big) = f(1, m-1) + 1.$$

Since $f(1,1) = 3$, so $f(1,2) = f(1,1) + 1 = 4$.

Similarly, we have $f(1,3) = 5$ and $f(1,4) = 6$. Finally,

$$f(2,2) = f\big(1, f(1,3)\big) = f(1,5)$$
$$= f(0+1,4+1) = f\big(0, f(1,4)\big) = f(0,6) = 7.$$

Further Reading

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